Reaction diffusion fronts with a free boundary: the speed of the advancing front

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Summary

The propagation of reaction diffusion fronts in a system with a free boundary is a topic of current interest since it constitutes a more accurate model of physical and biological problems where a clear position for the advancing front, be it a flame, population or temperature, can be observed. In contrast, in the classical Fisher type reaction diffusion problems the front extends to infinity for all $t > 0$. In this work we study a moving boundary with Stefan boundary conditions and use a variational approach to determine the speed of the advancing front. Our results are valid for general monostable and combustion reaction terms.

The problem

We study the reaction diffusion equation in one dimension with Stefan boundary conditions,

$$\begin{aligned} &u_t = u_{xx} + f(u) \text{ with } f(0) = f(1) = 0, \\ &u_x(0,t) = 0, \quad u(L(t),t) = 0, \quad \frac{dL(t)}{dt} = -\kappa u_x(L(t),t), \end{aligned} \tag{1}$$

where $L(t)$ is the free boundary and subscripts denote derivatives with respect to the independent variables x and t. The reaction term is either monostable or of combustion type. It has been shown that depending on the initial conditions $L(t = 0)$ and $u(x, 0)$ the perturbation may spread and gradually invade the space or may die away. In the spreading regime there is a unique traveling wave solution $u(x,t) = q(x - ct)$ [1,2]. The problem we address here is the determination of the speed of a spreading front as a function of the reaction term and of the Stefan parameter κ. Spreading traveling wave solutions $u(x,t) = q(x - ct)$ correspond to solutions of the ordinary differential equation

$$q_{zz} + cq_z + f(q) = 0, \text{ with } q(-\infty) = 1, \, q(0) = 0, \, q_z(0) = -c/\kappa$$

where $z = x - ct$.

We show that the speed c of the front can be obtained by means of a variational principle. In addition to provide a method to estimate the speed as accurately as desired, the variational principle allows to study the dependence of the speed on the parameters of the problem and to establish upper and lower bounds for the speed of general validity. For example, we show that

$$\frac{\kappa}{\kappa+1} c_{ZFK} \le c \le \frac{\kappa}{\kappa+1} c_0 \qquad where \qquad c_{ZFK} = \sqrt{2\int_0^1 f(u)du}$$

is the Zeldovich Frank-Kamenetskii speed and c_0 is the speed of the front in the classical case. It follows from this inequality that $\lim_{\kappa \to 0} c = 0$ and $\lim_{\kappa\to\infty} c = c_0$. Recent work [3] has considered the case of density dependent reaction, our results can be extended to this problem as well.

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References

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